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TECHNICAL NOTE 3139

TIME-DEPENDENT BUCKLING OF A UNIFORMLY HEATED COLUMN

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SUMMARY

A theoretical investigation is presented of the time-temperature-dependent buckling of a pin-jointed constant-section column, whose initial curvature is defined by a half-sine wave when the material is linearly viscoelastic and is heated uniformly along the column at a prescribed time rate. It was found that the deviations from straightness increase with time and become indefinitely large when heating reduces the Young's modulus of the material to the value at which the applied load is the Euler load of the column. When the column is heated very rapidly this critical time represents the limit of usefulness of the column. When heating takes place less rapidly the deflections of the column cause bending stresses exceeding the yield stress of the material at a time considerably smaller than the critical time. The equations presented permit the calculation of this reduction in the useful lifetime of the column.

INTRODUCTION

Analytical investigations and experiments have shown that a visco-elastic column at constant temperature subjected to a constant end load less than the Euler load will buckle if the load is maintained for a sufficiently long period of time. If the temperature is increased the time required for buckling decreases (refs. 1 to 3). This type of response of a structure to a sustained constant load is an example of the effect of creep, which is one of the factors responsible for the inelastic behavior of a column.

Creep buckling has been investigated recently in some detail by Kempner (refs. 4 and 5). In reference 4 the column was assumed to possess ideal linear viscosity of the Newtonian type. It was found that initial slight deviations of the center line of the column from the straight line increased continuously with time and became indefinitely large if the load and the temperature were maintained constant for an indefinitely long time. The mathematical concept of indefinitely large displacements is equivalent to the practical concept of buckling. However, a column becomes useless for practical purposes at an earlier time, namely when it becomes curved to such a degree that it cannot fulfill any more its structural purposes or when the bending stresses caused by large deviations from straightness cannot be supported by the material of the column. The situation is different when the material exhibits nonlinear viscosity. In that case, investigated by Kempner in reference 5, indefinitely large deformations are reached according to the equations derived in a finite rather than in an indefinitely large value of time. Nonlinearly viscoelastic columns buckle, therefore, with a snap action.

In all the work cited the column was assumed to be subjected to a constant load while the temperature was maintained at a constant level. In reality the temperature of the structure of a supersonic plane or guided missile is increased from the temperature prevailing at the airport or in the hangar to some elevated temperature corresponding to thermal equilibrium at full supersonic speed while the plane or missile is under the action of the loads existing in flight. It is of interest, therefore, to examine the problem of the buckling of a column subjected to a given constant load while its temperature is raised from ambient temperature on the ground to the thermal equilibrium temperature in supersonic flight. In the present report such an investigation is carried out with the assumption that heating takes place at a prescribed rate.

The inelastic behavior of a material was analyzed by the theory of mechanical models whereby the constituent phases of the material were replaced by mechanical models (ref. 6). The mechanical models are combinations of two model elements representing the two basic types of deformation:

(1) A perfectly elastic spring, which obeys Hooke's law, for elastic deformation.

(2) A dashpot, consisting of a perforated piston moving in a cylinder containing a viscous liquid, for viscous deformation.

The spring element is a model of a linearly elastic body while, if the liquid in the cylinder obeys Newton's law of viscosity, the dashpot is an example of a linearly viscous body. These basic elements may be coupled in series or parallel. When combined in series, a linear Maxwell model results and represents the model considered in this report to explain the inelastic behavior of the material.

The total strain of a Maxwell model under a uniaxial force consists of an elastic component (which contains Young's modulus) and a creep component (which contains a viscosity coefficient). In this report, Young's modulus and the viscosity coefficient are considered to be functions of temperature, while the temperature is a function of time. Inertia effects are neglected, since it is shown in reference 7 that there is good agreement between results of the dynamic and static approach for linearly viscoelastic columns.

Based on the assumption that during bending any transverse section, originally plane, remains plane and normal to the longitudinal fibers of the beam, the well-known fundamental relation between radius of curvature and strain is obtained. Assuming small deflections, the latter equation when differentiated with respect to time results in a third-order, linear partial differential equation. The assumption of a product solution in the independent variables reduces this equation to a first-order equation whose variables are separable.

The deflection of the column is expressed in terms of a simple, definite time integral containing Young's modulus and the viscosity coefficient. A solution is presented for a general linear variation of Young's modulus and viscosity coefficient with temperature and of temperature with time.

The author wishes to express his gratitude to Professor Joseph Kempner for his advice and to Professor N. J. Hoff, the supervisor of the project, for suggesting the problem.

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SYMBOLS

A	cross-sectional area of column
a_1,b_1	constants; coefficients in assumed linear Young's modulus - temperature relation
a_2,b_2	constants; coefficients in assumed linear viscosity-coefficient - temperature relation
a_3,b_3	constants; coefficients in assumed linear temperature-time relation
C	constant, E_oP/P_{E_o}
E	Young's modulus
E_a	activation energy
F	defined by equation (12)

F_i	maximum deflection of unloaded column
I	moment of inertia with respect to neutral axis
K	constant defined by equation (19)
L	length of column
M	moment of external forces
P	constant, compressive end load
$P_E(t)$	temperature (time) dependent Euler load, $\pi^2 E(t)I/L^2$
R	universal gas constant
T	temperature
T_a	temperature, deg abs
t	time
w	additional deflection from initially curved, unloaded column
w_i	initial deviation from straightness
w_o	elastic deflection resulting from instantaneous application of end load P
x	axial coordinate of column
z	normal distance from neutral surface
α	constant, a_2/b_2
β	constant, $(a_1 - C)/b_1$
ϵ	total axial strain
ϵ_c	creep component of axial strain
ϵ_e	elastic component of axial strain
λ	viscosity coefficient
ρ	radius of curvature
σ	stress

Φ defined by equation (13)

ψ function of temperature

Subscript:

o conditions at zero time

GENERAL ANALYSIS

The fundamental bending relation between the total strain ϵ and the radius of curvature ρ, which is based on the assumption that any transverse section, originally plane, remains plane and normal to the longitudinal fibers of the beam after bending, is

$$1/\rho = \epsilon/z \tag{1}$$

where z is the normal distance from the neutral surface.

The inelastic behavior of the column is expressed in terms of a linear Maxwell unit, that is, a series combination of a spring element obeying Hooke's law for linear elasticity and a dashpot whose fluid obeys Newton's law of viscosity for linear viscosity. The total strain of the Maxwell unit under a uniaxial force consists of the elastic component ϵ_e and the creep component ϵ_c (ref. 6), or

$$\epsilon = \epsilon_e + \epsilon_c = (\sigma/E) + \int_0^t (\sigma/\lambda)\ dt \tag{2}$$

where Young's modulus E and the viscosity coefficient λ are both functions of temperature, while the temperature is a function of time t.

The substitution of equation (2) into equation (1) and multiplication by z^2 yield

$$z^2/\rho = (\sigma z/E) + z\int_0^t (\sigma/\lambda)\ dt$$

Therefore

$$z^2 d(1/\rho)/dt = (\dot{\sigma}z/E) - (\dot{E}\sigma z/E^2) + (\sigma z/\lambda)$$

where $d/dt \equiv (\dot{\ })$. Integration of each term over the cross-sectional area A results in

$$\frac{d}{dt}(1/\rho)\int_A z^2\ dA = (1/E)\int_A \dot{\sigma}z\ dA - (\dot{E}/E^2)\int_A \sigma z\ dA + (1/\lambda)\int_A \sigma z\ dA \tag{3}$$

As the moment M of the external forces is expressed by

$$M = \int_A \sigma z \, dA$$

it follows that

$$\dot{M} = \int_A \dot{\sigma} z \, dA$$

Hence equation (3) reduces to

$$EI \, d(1/\rho)/dt = \dot{M} + \left[(E/\lambda) - (\dot{E}/E)\right]M \tag{4}$$

where I is the moment of inertia with respect to the neutral axis.

For small deflections

$$1/\rho = -d^2w/dx^2 \tag{5}$$

where w is the additional deflection from the initially curved, unloaded column and x is the axial coordinate (fig. 1). The initial displacement w_i of the pin-jointed, unloaded bar is assumed to be

$$w_i = F_i \sin(\pi x/L) \tag{6}$$

while the elastic deflection w_o resulting from the instantaneous application of the constant, compressive end load P is obtained from the solution of

$$E_oI \, d^2w_o/dx^2 = -M_o = -P(w_o + w_i) \tag{7}$$

The solution of equation (7) is

$$w_o = F_i \frac{P/P_{E_o}}{1 - \left(P/P_{E_o}\right)} \sin(\pi x/L) \tag{8}$$

where the subscript o refers to conditions at $t = 0$, and $P_{E_o} = \pi^2E_oI/L^2$ is the corresponding Euler load. When $t > 0$, the creep phenomenon must be considered; then the external moment M at any section is

$$M = P(w_i + w) \tag{9}$$

so that

$$\dot{M} = P\dot{w} \tag{10}$$

The substitution of equations (5), (9), and (10) into equation (4) and the rearrangement of terms result in the following third-order, linear partial differential equation:

$$EI \frac{\partial^3 w}{\partial t \partial x^2} + P\dot{w} + P\left[(E/\lambda) - (\dot{E}/E)\right]w = -P\left[(E/\lambda) - (\dot{E}/E)\right]w_i \tag{11}$$

Equation (11) must satisfy the conditions:

$$t = 0, \quad w = w_o$$

$$x = 0, L, \quad w = 0$$

As a solution, assume

$$w = F(t) \sin (\pi x/L) \tag{12}$$

which, when introduced into equation (11), together with equation (6), gives

$$-EI\dot{F}(\pi/L)^2 + P\dot{F} + P\left[(E/\lambda) - (\dot{E}/E)\right]F = -P\left[(E/\lambda) - (\dot{E}/E)\right]F_i$$

Let

$$P_E = P_E(t) = \pi^2 E(t) I/L^2$$

$$\Phi = \Phi(t) = \left[(E/\lambda) - (\dot{E}/E)\right]/(E - C) \tag{13}$$

where $C = E_o P / P_{E_o}$ is a constant; therefore the last equation becomes

$$\dot{F} - C\Phi F = C\Phi F_i$$

Hence

$$dF/(F + F_i) = C\Phi \, dt$$

whose solution is

$$F = (F_o + F_i)e^{C\int_0^t \Phi \, dt} - F_i$$

From equations (8) and (12), when $t = 0$

$$w = w_o$$

$$= F_i \frac{P/P_{E_o}}{1 - \left(P/P_{E_o}\right)} \sin(\pi x/L)$$

$$= F_o \sin(\pi x/L)$$

therefore

$$F_o = F_i \frac{P/P_{E_o}}{1 - \left(P/P_{E_o}\right)} = F_i C/(E_o - C)$$

and

$$F = F_i\left\{\left[E_o/(E_o - C)\right] e^{C\int_0^t \Phi \, dt} - 1\right\} \tag{14}$$

This expression is further simplified by the evaluation of the integral

$$\int_0^t \Phi \, dt = \int_0^t \frac{(E/\lambda) - (\dot{E}/E)}{(E - C)} dt$$

$$= \int_0^t \frac{E}{\lambda(E - C)} dt - \int_{E_o}^{E} \left[dE/E(E - C)\right]$$

$$= \int_0^t \frac{E}{\lambda(E - C)} dt - (1/C) \int_{E_o}^{E} \left\{\left[1/(E - C)\right] - (1/E)\right\} dE$$

$$= \int_0^t \frac{E}{\lambda(E - C)} dt - (1/C) \log_e \left[(E - C)E_o/(E_o - C)E\right]$$

Therefore

$$e^{C\int_0^t \Phi\,dt} = e^{C\int_0^t \frac{E}{\lambda(E-C)}dt - \log_e\left[(E-C)E_0/(E_0-C)E\right]}$$

$$= \left[E(E_0 - C)/E_0(E - C)\right] e^{C\int_0^t \frac{E}{\lambda(E-C)}dt}$$

Equation (14) becomes

$$F = F_i\left[\frac{e^{C\int_0^t \frac{E}{\lambda(E-C)}dt}}{1 - (C/E)} - 1\right]$$

But $C/E = E_0P/P_{E_0}E = P/P_E$; hence

$$F = F_i\left[\frac{e^{C\int_0^t \frac{E}{\lambda(E-C)}dt}}{1 - (P/P_E)} - 1\right] \tag{15}$$

and from equation (12)

$$w = F_i\left[\frac{e^{C\int_0^t \frac{E}{\lambda(E-C)}dt}}{1 - (P/P_E)} - 1\right] \sin(\pi x/L) \tag{16}$$

Equation (16) expresses the additional deflection w in terms of a simple, definite time integral containing Young's modulus $E(t)$ and the viscosity coefficient $\lambda(t)$. It will be noted that the deflected column retains its sine-curve characteristic and that the additional deflection tends to infinity in a finite time when the temperature (time) dependent Euler load $P_E(t)$ approaches the applied, constant end load P.

Although the equation indicates that the infinite deflection is independent of the viscosity coefficient, the time-dependent viscosity coefficient does, for practical purposes, influence the buckling failure if buckling failure is defined by some finite ratio of additional deflection to initial deflection.

APPLICATION OF GENERAL DISPLACEMENT EQUATION

Equation (16) is solved for the following assumed general linear relations between Young's modulus and temperature, viscosity coefficient and temperature, and temperature and time:

$$E = a_1 + b_1 T$$

$$\lambda = a_2 + b_2 T$$

$$T = a_3 + b_3 t$$

$$\int_0^t \frac{E}{\lambda(E - C)}\, dt = C\int_0^t \left[dt/\lambda(E - C)\right] + \int_0^t (dt/\lambda)$$

$$= \left(C/b_1 b_2 b_3\right)\int_{T_o}^{T} \left[dT/(T + \alpha)(T + \beta)\right] +$$

$$\left(1/b_2 b_3\right)\int_{T_o}^{T} \left[dT/(T + \alpha)\right]$$

where $\alpha = a_2/b_2$ and $\beta = (a_1 - C)/b_1$. But

$$\int_{T_o}^{T} \left[dT/(T + \alpha)(T + \beta)\right] = -\left[1/(\alpha - \beta)\right]\int_{T_o}^{T} \left[dT/(T + \alpha)\right] + \left[1/(\alpha - \beta)\right]\int_{T_o}^{T} \left[dT/(T + \beta)\right]$$

$$= -\left[\left[1/(\alpha - \beta)\right] \log_e (T + \alpha)\right]_{T_o}^{T} + \left[\left[1/(\alpha - \beta)\right] \log_e (T + \beta)\right]_{T_o}^{T}$$

$$= \left[1/(\alpha - \beta)\right] \log_e \frac{(T + \beta)(T_o + \alpha)}{(T_o + \beta)(T + \alpha)}$$

$$= \log_e \left[\frac{(T_o + \alpha)(T + \beta)}{(T + \alpha)(T_o + \beta)}\right]^{\frac{1}{\alpha-\beta}}$$

and

$$\int_{T_o}^{T} \left[dT/(T + \alpha)\right] = \log_e \left[(T + \alpha)/(T_o + \alpha)\right]$$

therefore

$$\int_0^t \frac{E}{\lambda(E - C)} dt = \left(C/b_1 b_2 b_3\right) \log_e \left[\frac{(T_o + \alpha)(T + \beta)}{(T + \alpha)(T_o + \beta)}\right]^{\frac{1}{\alpha-\beta}} +$$

$$\left(1/b_2 b_3\right) \log_e \left[(T + \alpha)/(T_o + \alpha)\right]$$

$$= \log_e \left\{\left[\frac{(T_o + \alpha)(T + \beta)}{(T + \alpha)(T_o + \beta)}\right]^{\frac{C}{b_1 b_2 b_3(\alpha-\beta)}} \left(\frac{T + \alpha}{T_o + \alpha}\right)^{\frac{1}{b_2 b_3}}\right\}$$

Let

$$\psi = \left\{\left[\frac{(T_o + \alpha)(T + \beta)}{(T + \alpha)(T_o + \beta)}\right]^{\frac{C}{b_1 b_2 b_3(\alpha-\beta)}} \left(\frac{T + \alpha}{T_o + \alpha}\right)^{\frac{1}{b_2 b_3}}\right\}$$

Then

$$e^{C\int_0^t \frac{E}{\lambda(E-C)}dt} = e^{C \log_e \psi}$$

$$= e^{\log_e \psi^C}$$

$$= \psi^C$$

$$= \left[\frac{(T_o + \alpha)(T + \beta)}{(T + \alpha)(T_o + \beta)}\right]^{\frac{C^2}{b_1 b_2 b_3(\alpha-\beta)}} \left(\frac{T + \alpha}{T_o + \alpha}\right)^{\frac{C}{b_2 b_3}}$$

$$= \left(\frac{T_o + \alpha}{T + \alpha}\right)^{\frac{C^2}{b_1 b_2 b_3(\alpha-\beta)} - \frac{C}{b_2 b_3}} \left(\frac{T + \beta}{T_o + \beta}\right)^{\frac{C^2}{b_1 b_2 b_3(\alpha-\beta)}} \tag{17}$$

Reference 8 presents an empirical quadratic formula for Young's modulus variation with temperature for 75S-T6 aluminum. The quadratic formula is assumed to be approximated by the following linear relation:

$$E = \left(10.5 \times 10^6\right) - \left(7.5 \times 10^3\ T\right) \tag{18}$$

where T is in degrees Fahrenheit.

In general, the viscosity-coefficient variation with temperature (ref. 9) is expected to be of the form

$$\lambda = Ke^{\frac{E_a}{RT_a}} \tag{19}$$

where

E_a	activation energy
R	universal gas constant
T_a	temperature, deg abs
K	constant

The approximation of a linear relation to an exponential curve is valid only for a narrow range of the temperatures. Based on the experimental curves of figure 3 in reference 2 and the use of equations (2) and (19), the following relation between viscosity coefficient and temperature is assumed:

$$\lambda = \left(6 \times 10^{13}\right) - \left(1.5 \times 10^{11}\ T\right) \tag{20}$$

It is assumed that at zero time the temperature of the column is 75° F and that its temperature rises to 400° F within 60 seconds. Therefore the temperature-time equation is

$$T = 75 + 5.42t \tag{21}$$

where t is the time in seconds.

A comparison of equations (18), (20), and (21) with the general linear equations gives

$$a_1 = 10.5 \times 10^6 \qquad b_1 = -7.5 \times 10^3$$

$$a_2 = 6 \times 10^{13} \qquad b_2 = -1.5 \times 10^{11}$$

$$a_3 = 75 \qquad b_3 = 5.42$$

If $P/P_{E_o} = 0.8$

$$C = E_o P/P_{E_o} = 9.9375 \times 10^6 \times 0.8 = 7.95 \times 10^6$$

$$\alpha = a_2/b_2 = -400$$

$$\beta = \left(a_1 - C\right)/b_1 = -340$$

$$\frac{C^2}{b_1 b_2 b_3(\alpha - \beta)} - \frac{C}{b_2 b_3} = -0.000163$$

$$\frac{C^2}{b_1 b_2 b_3(\alpha - \beta)} = -0.000173$$

Therefore equation (17) becomes

$$e^{C\int_0^t \frac{E}{\lambda(E-C)}dt} = \frac{(T_o + \beta)^{0.000173}(T + \alpha)^{0.000163}}{(T_o + \alpha)^{0.000163}(T + \beta)^{0.000173}} = \frac{|T - 400|^{0.000163}}{|T - 340|^{0.000173}}$$

since

$$\frac{(T_o + \beta)^{0.000173}}{(T_o + \alpha)^{0.000163}} = \frac{|75 - 340|^{0.000173}}{|75 - 400|^{0.000163}} \approx 1$$

Hence equation (16) is

$$\frac{w}{F_i} = \left\{\frac{|T - 400|^{0.000163}}{\left[1 - (C/E)\right]|T - 340|^{0.000173}} - 1\right\} \sin(\pi x/L) \qquad (22)$$

This relation is shown in figure 2 for the midpoint deflections of the column. Also shown in figure 2 is a curve for a column subjected to the following conditions:

$$P/P_{E_o} = 0.9$$

and

$$T = 75 + 0.542t$$

Of the two curves the steeper one corresponds to a rapid heating with a critical time of 48 seconds. The rate of heating corresponding to the flatter curve raises the temperature of the column in such a manner that the Euler load is reached in 244 seconds. Consequently, even the slow heating is rapid enough from the viewpoint of practical applications.

The diagram clearly indicates that indefinitely large deflections develop at the time when the increase in temperature lowers the Young's modulus of the material to such an extent that the applied load becomes the Euler load of the column. At the more rapid rate of heating presented the critical time can be considered in good approximation as the limit of practical usefulness of the column. On the other hand with the slow rate of heating the column becomes useless at a value of the time considerably smaller than the critical time. The cumulative effect of creep results in deflections of such magnitude that the linear stress-strain relationship loses its validity; the material starts to yield rapidly and the column buckles for all practical purposes before the theoretical critical time, based on idealized conditions, is reached. If a maximum deflection of the column amounting to 50 times the initial maximum deflection is considered inadmissible at the slow rate of heating, the column would become useless after 200 seconds rather than at the critical time of 244 seconds.

DISCUSSION

The general displacement equation (eq. (16)) indicates that the deflected column retains its sine-curve characteristic and that the additional deflection tends to infinity in a finite time when the temperature-dependent Euler load approaches the applied, constant end load.

Although the equation states that the infinite deflection is independent of the viscosity coefficient, the time-dependent viscosity coefficient does, for practical purposes, influence the buckling failure if buckling failure is defined by some finite ratio of additional deflection to initial deflection. This can be seen from figure 2 which gives the midpoint nondimensional additional deflections as a function of time for the assumed linear relations. At the more rapid rate of heating the time required for the temperature-dependent Euler load to attain the value of the applied constant end load is 48.8 seconds. With the slow rate of heating this critical time is 244 seconds. If 50 times the initial maximum deflection is considered as the limit of structural usefulness of the column the lifetime of the column is 200 seconds and not the theoretical critical time of 244 seconds.

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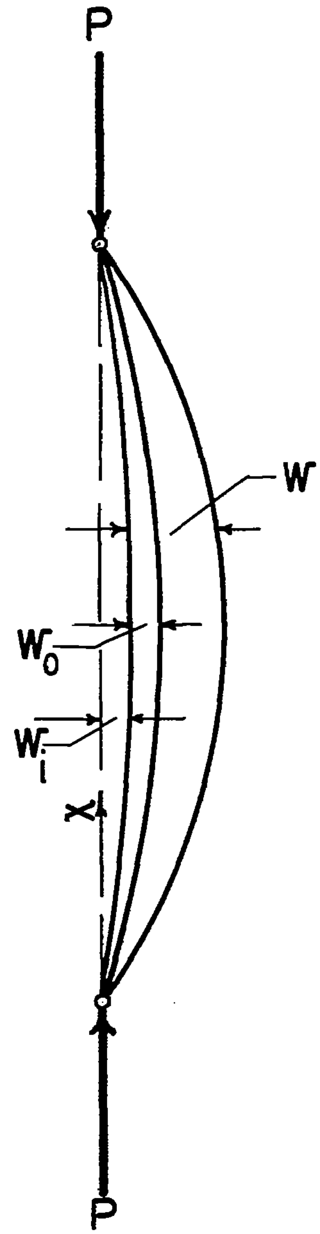

Figure 1.- Deflection notation for column.

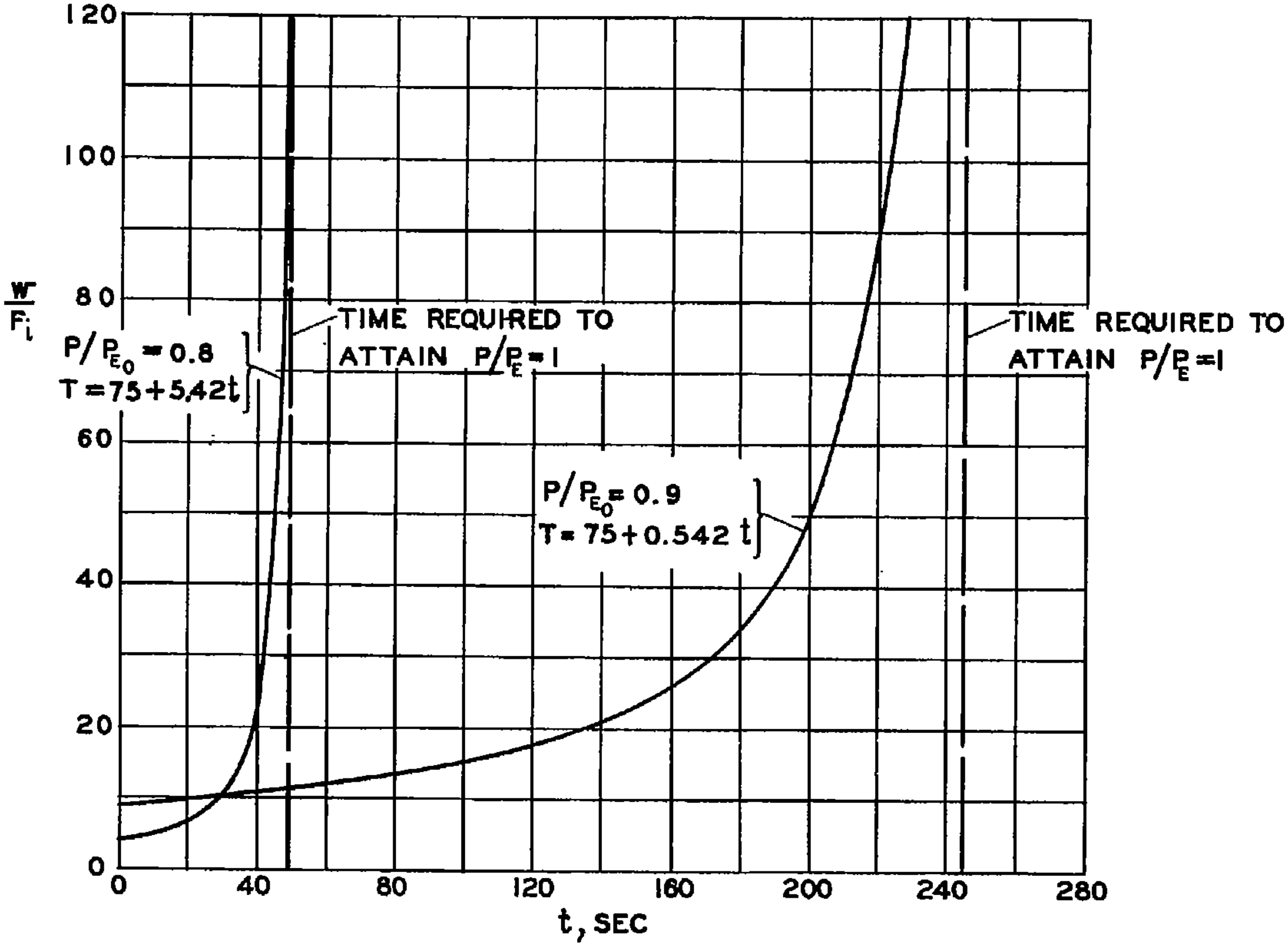

Figure 2.- Midpoint nondimensional additional deflections versus time.
Assumed relations: $E = (10.5 - 0.0075T)10^6$; $\lambda = (6 - 0.015T)10^{13}$; $x = L/2$.

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